Evolutionary Stability Against Multiple Mutations

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Abstract It is known (see e.g. Weibull (1995)) that ESS is not robust against multiple mutations. In this article, we introduce robustness against multiple mutations and study some equivalent formulations and consequences.

1 Introduction

The key concept in evolutionary game theory is ESS introduced by Maynard Smith and Price (1973) and early developments and applications to evolutionary biology are reported in Maynard Smith (1982). Some of the references to modern developments include Cressman (2003), Hofbauer and Sigmund (1998), Weibull (1995). ESS deals with the situation when there is only one rare mutation that can influence the population.

In practical scenarios, an incumbent strategy may be subjected to invasions by several mutations. As an example, we can consider bird nesting. During the season, birds search for a good location. To get the best location, they may need to compete with several others. In game theoretic terminology, this corresponds to the invasion of multiple mutations. Thus it is desirable to study the influence of multiple mutations. To the best of our knowledge, we are not aware of any studies which have dealt with the case of multiple mutations.

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In Weibull (1995) (and Vincent and Brown (2005)), it is noted that ESS is, in general, not robust against multiple mutations. One possible way to approach this issue may be to model this situation as a multi-player game and study the corresponding ESS. While there have been works dealing with ESS and multi-player games (see e.g. Broom et al. (1997)), we are not aware of any on the connection of this theory with multiple mutations. In this work, we take alternative approach, where we extend the notion of ESS to take care of the multiple mutations. We provide some interesting consequences. Our approach leads to useful refinement of Nash equilibrium.

Our article is structured as follows. After the introductory section, we introduce evolutionary stability against multiple mutations in Section 2. We show that evolutionary stability against multiple mutations is equivalent to evolutionary stability against two mutations in a special case. In Section 3, we provide an equivalent formulation. Using the ideas in this equivalent formulation, we show that evolutionary stability against multiple mutations is equivalent to evolutionary stability against two mutations. Section 4 introduces the concept of local dominance and discusses its connections with evolutionary stability. It also draws differences with strict Nash equilibrium. In Section 5, we establish the fact that evolutionarily stable strategy against multiple mutations is necessarily a pure strategy, a property shared by strict Nash equilibrium. One consequence of this fact is the existence of uniform invasion barrier. We characterize the evolutionarily stable strategies against multiple mutations in 2×2 games. We conclude our article with some comments and directions for further research in Section 6.

2 Evolutionary Stability

We consider symmetric games with payoff function $u : \Delta \times \Delta \to \mathbb{R}$, where Δ is probability simplex in $\mathbb{R}^k$ and u is given by the affine function

$$u(p, q) = \sum_{i,j=1}^{k} p_i q_j u(e^i, e^j).$$

Here $e^1 = (1, 0, 0, \cdots, 0), \cdots, e^k = (0, \cdots, 0, 1) \in \mathbb{R}^k$ denote the pure strategies of the players. We, first recall the definition of an evolutionarily stable strategy (ESS for short).

Definition 2.1 A strategy $p \in \Delta$ is called ESS, if for any mutant strategy $r \neq p$, there is an invasion barrier $\epsilon(r) \in (0, 1)$ such that

$$u(p, \epsilon r + (1 - \epsilon)p) > u(r, \epsilon r + (1 - \epsilon)p) \text{ for all } 0 < \epsilon \leq \epsilon(r). \tag{1}$$

We gather some notations that we use in due course:

$$BR(p) = \{q \in \Delta \ : \ u(q, p) \geq u(r, p) \ \ \forall r \in \Delta\},$$
$$\Delta^{NE} = \{p \in \Delta \ : \ p \in BR(p)\}.$$

By definition, an ESS is robust against any single mutation r appearing in small proportions. A natural question that arises is that whether an ESS is robust

against multiple mutations. It is known (see e.g. Weibull (1995)) that an ESS may not be robust against multiple mutations. We now provide an example to illustrate this fact.

Example 2.1 Consider the 2×2 symmetric game with fitness (or, payoff) matrix $U = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. The unique ESS of this game is $p = (\frac{1}{2}, \frac{1}{2})$.

Consider $r^1 = (\frac{1}{4}, \frac{3}{4})$ and $r^2 = (\frac{3}{4}, \frac{1}{4})$. Now, for any $0 < \epsilon < \frac{1}{2}$,

$$u(p, \epsilon r^1 + \epsilon r^2 + (1 - 2\epsilon)p) = -\frac{1}{2}.$$

But

$$u(r_1, \epsilon r^1 + \epsilon r^2 + (1 - 2\epsilon)p) = -\left[\epsilon(\frac{10}{16} + \frac{6}{16}) + (1 - 2\epsilon)\frac{1}{2} \right] = -\frac{1}{2}.$$

Thus p is not robust against simultaneous mutations r^1, r^2, whenever they appear in equal propositions.

The above example makes it clear that the Definition 2.1 is inadequate to capture the robustness or evolutionary stability against multiple mutations. This motivates the following definition.

Definition 2.2 Let m be a positive integer. A strategy $p \in \Delta$ is said to be evolutionarily stable (or robust) against 'm' mutations if, for every $r^1, \cdots, r^m \neq p$, there exists $\bar{\epsilon} = \bar{\epsilon}(r^1, \cdots, r^m) \in (0, 1)$ such that

$$u(p, \epsilon_1 r^1 + \cdots + \epsilon_m r^m + (1 - \epsilon_1 - \cdots - \epsilon_j)p)$$
$$> \max_{1 \leq i \leq m} u(r^i, \epsilon_1 r^1 + \cdots + \epsilon_j r^j + (1 - \epsilon_1 - \cdots - \epsilon_j)p),$$

for all $\epsilon_1, \ldots, \epsilon_j \in (0, \bar{\epsilon}]$.

Remark 2.1 Clearly if $m = 1$, then the above definition coincides with the definition of ESS.

Definition 2.3 If $\bar{\epsilon}$ in Definition 2.2 can be chosen independent of $(r^1, r^2, \cdots, r^m)$, then we refer to $\bar{\epsilon}$ as a *uniform invasion barrier* for p corresponding to m mutations.

Remark 2.2 The uniform invasion barrier $\bar{\epsilon}$, in general, depends on m. However, we can choose a bound on the total fraction i.e., $\epsilon_1 + \epsilon_2 + \cdots + \epsilon_m$ of the m mutations can be taken to be independent of m. We come back to this point later.

We now provide a surprising result albeit with a very simple proof.

Theorem 2.1 *Let $m > 2$ and $p \in \Delta$. If the strategy p is evolutionarily stable against two mutations with uniform invasion barrier, then it is evolutionarily stable against m mutations with uniform invasion barrier.*

Proof Let p be evolutionarily stable against two mutations with uniform invasion barrier $\bar{\epsilon}$. Let $r^1, r^2, \cdots, r^m$ be m strategies different from p.

Let $\epsilon_i \in (0, \frac{\bar{\epsilon}}{m}]$, $i = 1, 2, \cdots, m$. Consider

$$
\begin{aligned}
w &= \epsilon_1 r^1 + \epsilon_2 r^2 + \cdots + \epsilon_m r^m + (1 - \epsilon_1 - \epsilon_2 - \cdots - \epsilon_m)p \\
&= (\epsilon_1 + \cdots + \epsilon_{m-1})s + \epsilon_m r^m + (1 - \epsilon_1 - \epsilon_2 - \cdots - \epsilon_m)p,
\end{aligned}
$$

where

$$
s = \frac{\epsilon_1}{\epsilon_1 + \cdots + \epsilon_{m-1}} r^1 + \cdots + \frac{\epsilon_{m-1}}{\epsilon_1 + \cdots + \epsilon_{m-1}} r^{m-1}.
$$

Obviously $(\epsilon_1 + \cdots + \epsilon_{m-1}), \epsilon_m \leq \bar{\epsilon}$. Now considering s and r^m as mutations, we have

$$
u(p, w) > \max\{u(s, w), u(r^m, w)\},
$$

since p is robust against two mutations with uniform invasion barrier $\bar{\epsilon}$. Thus

$$
u(p, w) > u(r^m, w).
$$

Instead of r^m, we can use any r^j, $j = 1, 2, \cdots, m - 1$ in the above analysis which leads to

$$
u(p, w) > u(r^j, w)
$$

for every $j = 1, 2, \cdots, m$. Thus p is evolutionarily stable against m mutations and $\frac{\bar{\epsilon}}{m}$ is uniform invasion barrier corresponding to m mutations. $\square$

Remark 2.3 The theorem assumes the existence of uniform invasion barrier. This is, indeed, the case as we show later.

3 An Equivalent Formulation

In this section, we provide an equivalent formulation for the evolutionary stability against two mutations.

Theorem 3.1 *For $p \in \Delta$, the following are equivalent:*

(a) *p is robust against two mutations;*
(b) *$p \in \Delta^{NE}$, and, for every $q \in BR(p) \setminus \{p\}$ and $r \in \Delta$,*

$$
u(p, q) > u(q, q) \quad and \quad u(p, r) \geq u(q, r).
$$

Proof We start with (a) $\Rightarrow$ (b). Assume that p is robust against two mutations. In particular p is an ESS. Let $q \in BR(p) \setminus \{p\}$ and $r \in \Delta$. For small enough $\epsilon_1, \epsilon_2 > 0$, we must have

$$
u(p, (1 - \epsilon_1 - \epsilon_2)p + \epsilon_1 q + \epsilon_2 r) > u(q, (1 - \epsilon_1 - \epsilon_2)p + \epsilon_1 q + \epsilon_2 r).
$$

Rearranging the terms, we get

$$
\epsilon_1\{u(p, q) - u(q, q)\} + \epsilon_2\{u(p, r) - u(q, r)\} + (1 - \epsilon_1 - \epsilon_2)\{u(p, p) - u(q, p)\} > 0.
$$

Since $q \in BR(p)$, the third term is zero and hence, for small enough $\epsilon_1, \epsilon_2 > 0$, we have

$$
\epsilon_1[u(p, q) - u(q, q)] + \epsilon_2[u(p, r) - u(q, r)] > 0.
$$

Since p is an ESS and $q \in BR(p) \setminus \{p\}$, we have $u(p,q) > u(q,q)$. From this and the above inequality, it follows that $u(p,r) \geq u(q,r)$.

We now show that (b) $\Rightarrow$ (a). Assume (b). Let the mutations r^1, r^2 appear in proportions ϵ_1, ϵ_2 respectively. For $i = 1, 2$, let

$$h_i(\epsilon_1, \epsilon_2) := u(p, \epsilon_1 r^1 + \epsilon_2 r^2 + (1 - \epsilon_1 - \epsilon_2)p) - u(r_i, \epsilon_1 r^1 + \epsilon_2 r^2 + (1 - \epsilon_1 - \epsilon_2)p)$$

We need to show that for ϵ_1, ϵ_2 small enough, $h_i(\epsilon_1, \epsilon_2) > 0$ for each $i = 1, 2$. Note that

$$\begin{aligned} h_i(\epsilon_1, \epsilon_2) = \epsilon_1[u(p, r^1) - u(r_i, r^1)] \\ + \epsilon_2[u(p, r^2) - u(r^i, r^2)] + (1 - \epsilon_1 - \epsilon_2)[u(p, p) - u(r^i, p)]. \quad (2) \end{aligned}$$

Fix i. If $r^i \in BR(p)$, then the third term on the R.H.S. of (2) is zero. By hypothesis, $u(r^i, r^i) < u(p, r^i)$ and $u(r^i, r^j) \leq u(p, r^j)$, for $j \neq i$. Therefore, for $\epsilon_1, \epsilon_2 > 0$, $h_i(\epsilon_1, \epsilon_2) > 0$ whenever $r^i \in BR(p)$.

Now let $r^i \notin BR(p)$. Then $u(p, p) - u(r^i, p) > 0$. Hence for sufficiently small ϵ_1 and ϵ_2, we must have $h(\epsilon_1, \epsilon_2) > 0$. Thus p is robust against two mutations. $\square$

Remark 3.1 The above characterization suggests the following interpretation of evolutionary stability against multiple mutations: An ESS is robust against multiple mutations if and only if it dominates all strategies that are best responses to it.

A careful observation of the proof of Theorem 3.1 shows that evolutionary stability against 2 mutations is equivalent to evolutionary stability against any m mutations, $m \geq 2$. We now prove this equivalence. Note that in the previous section, we showed this result when there is uniform invasion barrier.

Theorem 3.2 *A strategy is evolutionarily stable against two mutations if and only if it is evolutionarily stable against m mutations, where $m > 2$.*

Proof We will only show that evolutionary stability against two mutations implies the evolutionary stability against m mutations, the other part being trivial.

Let p be evolutionarily stable against two mutations. Let $r^1, r^2, \cdots, r^m$ be m mutations that appear with proportions $\epsilon_1, \epsilon_2, \cdots, \epsilon_m$, respectively. For $i = 1, 2, \cdots, m$, let

$$\begin{aligned} h_i(\epsilon_1, \epsilon_2, \cdots, \epsilon_m) := u(p, \epsilon_1 r^1 + \epsilon_2 r^2 + \cdots + \epsilon_m r^m + (1 - \epsilon_1 - \epsilon_2 - \cdots - \epsilon_m)p) \\ - u(r_i, \epsilon_1 r^1 + \epsilon_2 r^2 + \cdots + \epsilon_m r^m + (1 - \epsilon_1 - \epsilon_2 - \cdots - \epsilon_m)p) \end{aligned}$$

We need to show that for $\epsilon_1, \epsilon_2, \cdots, \epsilon_m$ small enough, $h_i(\epsilon_1, \epsilon_2, \cdots, \epsilon_m) > 0$ for each $i = 1, 2, \cdots, m$. Note that

$$\begin{aligned} h_i(\epsilon_1, \epsilon_2, \cdots, \epsilon_m) = \epsilon_1[u(p, r^1) - u(r^i, r^1)] + \epsilon_2[u(p, r^2) - u(r^i, r^2)] \\ + \cdots + \epsilon_m[u(p, r^m) - u(r^i, r^m)] \\ + (1 - \epsilon_1 - \epsilon_2 - \cdots - \epsilon_m)[u(p, p) - u(r^i, p)]. \quad (3) \end{aligned}$$

Fix i. If $r^i \in BR(p)$, then $u(r^i, p) - u(p, p) = 0$. From Theorem 3.1, we have

$$u(r^i, r^i) < u(p, r^i) \text{ and } u(r^i, r^j) \leq u(p, r^j)$$

for all $j \neq i$. As a result, we have $h_i(\epsilon_1, \epsilon_2, \cdots, \epsilon_m) > 0$ for $\epsilon_1, \epsilon_2, \cdots, \epsilon_m > 0$, whenever $r^i \in BR(p)$.

Now let $r^i \notin BR(p)$. Then $u(p,p) - u(r^i,p) > 0$. Thus for sufficiently small $\epsilon_1, \epsilon_2, \cdots, \epsilon_m > 0$, we must have $h(\epsilon_1, \epsilon_2, \cdots, \epsilon_m) > 0$. And hence p is evolutionarily stable against m mutations.   □

Remark 3.2 As a result of the Theorem 3.2, if a strategy is evolutionarily stable against $m \geq 2$ mutations, we refer to it simply as evolutionarily stable against multiple mutations, by suppressing the number m.

4 Local Dominance

In evolutionary game theory, ESS is characterized by means of two notions: uniform invasion barrier and local superiority. Uniform invasion barrier is already introduced. Local superiority of a mixed strategy p implies that $u(p,q) > u(q,q)$ for every $q \neq p$ in a neighborhood of p. The interpretation of this notion is as follows: p is ESS if and only if in a neighborhood of p, there can not be any other symmetric Nash equilibrium other than p. We now introduce the corresponding generalization of local superiority to the case of multiple mutations.

Definition 4.1 (Local Dominance) A strategy $p \in \Delta$ is said to be locally dominant if there is a neighborhood U of p such that $u(p,r) \geq u(s,r)$ and $u(p,r) > u(r,r)$ for every $s, r \in U \setminus \{p\}$.

Remark 4.1 Note that if p is locally dominant, then we can easily show that

$$u(p,r) \geq u(s,r) \text{ and } u(p,r) > u(r,r)$$

for every $s \in \Delta$ and $r \in U \setminus \{p\}$, where U is the neighborhood as in the definition of local dominance.

We now show that evolutionary stability against multiple mutations and local dominance are equivalent.

Theorem 4.1 *A strategy p is evolutionarily stable against multiple mutations if and only if it is locally dominant.*

Proof Assume that p is locally dominant. By definition, p is an ESS. Let $q \in BR(p)$ and $q, r \neq p$. To show that p is robust against multiple mutations, it suffices to show that $u(p,r) \geq u(q,r)$.

Note that $r^\epsilon = \epsilon r + (1-\epsilon)p$ is close to p whenever $\epsilon > 0$ is small enough. Since p is weak locally dominant, for $\epsilon > 0$ small enough, we must have

$$0 \leq u(p, r^\epsilon) - u(q, r^\epsilon) = \epsilon[u(p,r) - (q,r)].$$

This implies that $u(p,r) \geq u(q,r)$.

Now assume that p is robust against multiple mutations. Let $s \neq p$. We first show that there exists a neighborhood $V = V(s)$ of p such that

$$f(r) := u(p,r) - u(s,r) \geq 0 \tag{4}$$

for all $r \in V \setminus \{p\}$.

Now

$$f(e_\epsilon^i) = \epsilon[u(p, e^i) - u(s, e^i)] + (1 - \epsilon)[u(p, p) - u(s, p)],$$

where $e_\epsilon^i = \epsilon e^i + (1 - \epsilon)p$.

If $s \in BR(p)$, then, by hypothesis, $f(e_\epsilon^i) \geq 0$ for every $0 \leq \epsilon \leq 1$. If $s \notin BR(p)$, then clearly there exists $\bar{\epsilon}_i(s) \in (0, 1)$ such that $f(e_\epsilon^i) > 0$ for $0 \leq \epsilon < \bar{\epsilon}_i(s)$.

Thus $f(r) \geq 0$ when $r \in L$;

$$L = \{w \in \Delta \ : \ w = \epsilon e^i + (1 - \epsilon)p \ \text{ for some } 1 \leq i \leq k, \ 0 \leq \epsilon < \min_{1 \leq i \leq k} \bar{\epsilon}_i(s)\}.$$

This clearly implies that $f(r) \geq 0$ for every r in the convex hull $V = V(s)$ (which is also a neighborhood of p) of L. Therefore $u(p, r) \geq u(s, r)$ for every s and $r \in U := \cap_{i=1}^{k} V(e^i)$. This implies that p is locally dominant. □

In the following proposition we show that the inequality in the local dominance is strict for all s whose support has a non-empty intersection with the support of p.

Theorem 4.2 *Let p be robust against multiple mutations. Then there is a neighborhood U such that $u(p, r) > u(s, r)$ for all $r \in U$ and $s \in U$ such that $\mathbf{supp}(s) \cap \mathbf{supp}(p) = \emptyset$.*

Proof Let p be evolutionarily stable against multiple mutations and let

$$C = \{p \in \Delta : p_i = 0 \text{ for some } i \in \mathbf{supp}(p)\}.$$

Clearly C is compact and $p \in C$. We can choose $\bar{\epsilon} > 0$ such that

$$u(p, \epsilon_1 r + \epsilon_2 s + (1 - \epsilon_1 + \epsilon_2)p) > u(r, \epsilon_1 r + \epsilon_2 s + (1 - \epsilon_1 - \epsilon_2)p)$$

for all $r, s \in C$ and $0 < \epsilon_1, \epsilon_2 \leq \bar{\epsilon}$. Hence

$$u(p, \epsilon_1 r + \epsilon_2 s + (1 - \epsilon_1 + \epsilon_2)p) > u(\alpha_1 r + \alpha_2 s + (1 - \alpha_1 + \alpha_2)p, \epsilon_1 r + \epsilon_2 s + (1 - \epsilon_1 + \epsilon_2)p)$$

for every $0 < \alpha_1, \alpha_2, \epsilon_1, \epsilon_2 < \bar{\epsilon}$. If we choose $U = B(p; \bar{\epsilon})$, then from the above we have

$$u(p, r) > u(s, r)$$

for all $r, s \neq p \in U$. □

We now make a definition.

Definition 4.2 (Strict Local Dominance) A strategy $p \in \Delta$ is said to be strictly locally dominant if there is a neighborhood U of p such that $u(p, r) > u(s, r)$ for every $s, r \in U \setminus \{p\}$.

A strict Nash equilibrium is always strictly locally dominant. We may think that the other way is also correct. However it is not the case as the following example shows.

Example 4.1 Consider the 2×2 symmetric game with fitness matrix

$$U = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Clearly $BR(e^2) = \Delta$, and hence it is not a strict symmetric Nash equilibrium. But e^2 is an ESS, since for any $q \neq e^2$,

$$u(q,q) = -q_1^2 < 0 = u(e^2, q).$$

Furthermore, for $q, r \neq e^2$,

$$u(q,r) = -q_1 r_1 < 0 = u(e^2, r).$$

Therefore, by Theorem 3.1, e^2 is robust against m mutations, for any $m \geq 1$.

5 Pure Strategies and Uniform Invasion Barrier

An ESS can be mixed. On contrary, evolutionarily stable strategy against multiple mutations is always pure. We now prove this fact. Note that a strict Nash equilibrium is also necessarily pure.

Theorem 5.1 *An evolutionarily stable strategy against multiple mutations is necessarily a pure strategy.*

Proof Let p be evolutionarily stable against multiple mutations. If possible, let p be not a pure strategy. Let $p = (p_1, p_2, \cdots, p_k)$. Let $\bar{\epsilon} = \bar{\epsilon}(e^1, e^2, \cdots, e^k)$ be the invasion barrier corresponding to all the k pure mutations. Let $r = \alpha_1 e^1 + \alpha_2 e^2 + \cdots + \alpha_k e^k + (1 - \alpha_1 - \alpha_2 - \cdots - \alpha_k)p$, where $0 < \alpha_1, \alpha_2, \cdots, \alpha_k < \bar{\epsilon}$. Then, we have

$$\sum_{i=1}^{k} p_i u(e^i, r) = u(p, r) > \max\{u(e^1, r), u(e^2, r), \cdots, u(e^k, r)\}, \qquad (5)$$

which is a contradiction. Thus p must be pure.   □

So far we have not answered the question of existence of uniform invasion barrier. We now answer this question.

Theorem 5.2 *If p is robust against multiple mutations, then it has uniform invasion barrier.*

Proof Let p be robust against multiple mutations. Then p is necessarily pure. Without loss of generality, let us assume that $p = e^k$.

Let $\bar{\epsilon}$ be the invasion barrier corresponding to the pure strategies $e^1, \cdots, e^{k-1}$. We show that $\frac{\bar{\epsilon}}{m}$ is invasion barrier for any m mutations with $m \geq k - 1$.

Let $r^1, r^2, \cdots, r^m$ be m mutations with proportions $\epsilon_1, \epsilon_2, \cdots, \epsilon_m$ respectively. Choose α_i^j, $i = 1, 2, \cdots, m$, $j = 1, 2, \cdots, k$ such that $r^i = \alpha_i^1 e^1 + \alpha_i^2 e^2 + \cdots + \alpha_i^k e^k$. Consder

$$w = \epsilon_1 r^1 + \epsilon_2 r^2 + \cdots + \epsilon_m r^m - (1 - \epsilon_1 - \epsilon_2 - \cdots - \epsilon_m)p$$
$$= \beta_1 e^1 + \beta_2 e^2 + \cdots + \beta_k e^k + (1 - \beta_1 - \beta_2 - \cdots - \beta_k)p$$
$$= \beta_1 e^1 + \beta_2 e^2 + \cdots + \beta_{k-1} e^{k-1} + (1 - \beta_1 - \beta_2 - \cdots - \beta_{k-1})p$$

where
$$\beta_i = \epsilon_1 \alpha_1^i + \epsilon_2 \alpha_2^i + \cdots + \epsilon_m \alpha_m^i \text{ and } i = 1, 2, \cdots, m.$$

If we choose $\epsilon_1, \epsilon_2, \cdots, \epsilon_m \leq \frac{\bar{\epsilon}}{m}$, then from the definition of evolutionary stability we have,
$$u(p, w) > u(e^j, w), j = 1, 2, \cdots, k - 1.$$

Thus for any i, $i = 1, 2, \cdots, m$, we have

$$u(p, w) = \sum_{j=1}^{k} \alpha_i^j u(p, w) > \sum_{j=1}^{k} \alpha_i^j u(e^j, w) = u(r^i, w).$$

Here we have used the above $k - 1$ inequalities together with the fact that $p = e^k$. Thus p is evolutionarily stable against m mutations with $\frac{\bar{\epsilon}}{m}$ as the uniform invasion barrier.

Note that any invasion barrier corresponding to m mutations is also invasion barrier corresponding to n mutations, where $n < m$. This completes the proof the thoerem. □

Remark 5.1 As a consequence of the proof, we note that the bound on the total fraction of the m mutations $\epsilon_1 + \epsilon_2 + \cdots + \epsilon_m$ can be chosen to be $\bar{\epsilon}$, which is independent of m.

A careful observation of the proof of the Theorem 5.2 gives a complete characterization of evolutionary stability against multiple mutations in 2×2 games. We omit the proof as it is essentially contained in the proof of the Theorem 5.2.

Theorem 5.3 *For two player games, a pure strategy p is evolutionarily stabile against multiple mutations if and only if it is ESS.*

Remark 5.2 We believe that this result is not true for games with three or more strategies. However, we neither have a proof nor a counter example.

6 Conclusions

In this article, we introduced and studied the evolutionary stability against multiple mutations. We showed that the number of mutations ($m \geq 2$) is invariant. Further the evolutionarily stable strategy against multiple mutations is necessarily a pure strategy. This notion coincides with ESS in the case of 2×2 symmetric games, as long as the ESS is pure. Like in the case of ESS, we do not have any general result on the existence. Again it is in general non-unique. In deed, strict Nash equilibrium, itself, can be non-unique.

Our study also leaves several question to explore further. Firstly, note that classical Hawk-Dove game does not have any evolutionarily stable strategy against multiple mutations. We do not know if this has any implication in evolutionary biology as of now.

In 2×2 case, an ESS is evolutionarily stable against multiple mutations if and only if the ESS is pure. We believe that this result is not true in general case. However, we do not have any counter example.

Whether evolutionary stability against multiple mutations can be seen as a concept related to multiplayer games seems to be an interesting issue to be explored. If such a connection can be drawn, we can, hopefully, apply our results in situations modeled as multiplayer games e.g., in bird nesting.

References

Broom, M., Canning, C., Vickers, G., 1997. Multi-player matrix games. Bulletin of Math. Bilogoy 59, 931 – 952.

Cressman, R., 2003. Evolutionary Dynamics and Extensive Form Games. The MIT Press.

Hofbauer, J., Sigmund, K., 1998. Evolutionary Games and Population Dynamics. Cambridge University Press.

Maynard Smith, J., 1982. Evolution and the Theory of Games. Cambridge University Press.

Maynard Smith, J., Price, G. R., 1973. The logic of animal conflict. Nature 246, 15 – 18.

Vincent, T. L., Brown, J. S., 2005. Evolutionary Game theory, Natural Selection, Darwinian Dynamics. Cambridge University Press.

Weibull, J. W., 1995. Evolutionary Game Theory. MIT Press.